

# On minimal triangle-free planar graphs with prescribed 1-defective chromatic number

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## Abstract

A graph is  $(m, k)$ -colourable if its vertices can be coloured with  $m$  colours such that the maximum degree of the subgraph induced on the set of all vertices receiving the same colour is at most  $k$ . The  $k$ -defective chromatic number  $\chi_k(G)$  is the least positive integer  $m$  for which graph  $G$  is  $(m, k)$ -colourable. Let  $f(m, k; \text{planar})$  be the smallest order of a triangle-free planar graph such that  $\chi_k(G) = m$ . In this paper we show that  $f(3, 1; \text{planar}) = 11$ .

**Keywords.**  $k$ -defective chromatic number  $k$ -independence triangle-free graph  $(3, 1)$ -critical graph planar graph

**Math Review Codes.** MSC 05C15 MSC 05C35

## 1 Introduction

In this paper we consider undirected graphs with no loops or multiple edges. For all undefined concepts and terminology we refer to [4].

If  $u$  is a vertex of  $G$  then  $d_G(u)$ ,  $N_G(u)$  and  $N_G[u]$  denote respectively the degree, the neighbourhood, and the closed neighbourhood of  $u$  in  $G$ . Let  $\varepsilon(G)$  denote the number of edges in  $G$ .

Let  $k$  be a nonnegative integer. A subset  $U$  of  $V(G)$  is  $k$ -independent if  $\Delta(G[U]) \leq k$ . A 0-independent set is an independent set in the usual sense. A graph  $G$  is  $(m, k)$ -colourable if it is possible to assign  $m$  colours, say  $1, 2, \dots, m$  to the vertices of  $G$ , one colour to each vertex, such that the set of all vertices receiving the same colour is  $k$ -independent. The smallest

integer  $m$  for which  $G$  is  $(m, k)$ -colourable is called the  *$k$ -defective chromatic number* of  $G$  and is denoted by  $\chi_k(G)$ . A graph  $G$  is said to be  *$(m, k)$ -critical* if  $\chi_k(G) = m$  and  $\chi_k(G - u) < m$  for every  $u$  in  $V(G)$ . A graph  $G$  is said to be  *$(m, k)$ -edge-critical* if  $\chi_k(G) = m$  and  $\chi_k(G - e) < m$  for every  $e$  in  $E(G)$ .

It is easy to see that the following statements are equivalent.

- (i)  $G$  is  $(m, k)$ -colourable.
- (ii) There exists a partition of  $V(G)$  into  $m$  sets each of which is  $k$ -independent.
- (iii)  $\chi_k(G) \leq m$ .

Clearly  $\chi_0(G)$  is the usual chromatic number. It is easy to see that  $\chi_k(G) \leq \lceil \frac{|V(G)|}{k+1} \rceil$  where  $|V(G)|$  is the order of  $G$ . Furthermore,  $\chi_k(G) \leq \chi_{k-1}(G)$  for all integers  $k \geq 1$ .

The concept of  $k$ -defective chromatic number has been extensively studied in the literature (see [2, 6, 7, 8, 9, 13, 17, 19]). Given a positive integer  $m$ , there exists a triangle-free graph with  $G$  with  $\chi_k(G) = m$ . A natural question that arises is: what is the smallest order of a triangle-free graph  $G$  with  $\chi_k(G) = m$ ? We denote this smallest order by  $f(m, k)$ . The parameter  $f(m, 0)$  has been studied by several authors (see [3, 5, 10, 14, 12]) and  $f(m, 0)$  is determined for  $m \leq 5$ . It has also been shown that  $f(3, k) \leq 4k+5$  for all  $k \geq 0$ ;  $f(3, 1) = 9$  and  $f(3, 2) = 13$  (see [17, 2]). In the same papers the corresponding extremal graphs have been characterized. In a recent paper [1] the authors characterized triangle-free graphs on 10 vertices with  $\chi_1(G) = 3$ . They proved that every triangle-free graph  $G$  of order 10 with  $\chi_1(G) = 3$ , except one, contains one of the four (3,1)-critical triangle-free graphs of order 9. Furthermore, the exceptional triangle-free graph on 10 vertices is a (3,1)-edge-critical graph.

Grötzsch [11] proved that if  $G$  is a triangle-free planar graph then  $\chi_0(G) \leq 3$ . Hence  $\chi_k(G) \leq 3$  for any triangle-free planar graph  $G$  and  $k \geq 1$ .

We define  $f(m, k; \text{planar})$  to be the smallest order of a triangle-free planar graph  $G$  with  $\chi_k(G) = m$ . Note that  $f(m, k; \text{planar}) \geq f(m, k)$ . The problem of determining  $f(m, k; \text{planar})$  is relevant only for  $m = 2$  and  $m = 3$ . Considering the bipartite graph  $K_{2,k}$  it is easy to see that  $f(2, k; \text{planar}) = k + 2$ . Clearly  $f(3, 0; \text{planar}) = 5$ . In this paper we prove that  $f(3, 1; \text{planar}) = 11$ .

In all the figures in this paper a double line between sets  $X$  and  $Y$  means that every vertex of  $X$  is adjacent to every vertex of  $Y$ .

## 2 Preliminary results

We need the following results, proofs of the theorems being in the papers cited.

**Theorem 2.1** ([13, 15]) *For any graph  $G$ ,*

$$\chi_k(G) \leq \lceil \frac{\Delta(G) + 1}{k + 1} \rceil = 1 + \lfloor \frac{\Delta(G)}{k + 1} \rfloor.$$

**Theorem 2.2** ([8]) *If  $G$  is a  $(\alpha, k)$ -critical graph then  $\delta(G) \geq \alpha - 1$ .*

**Theorem 2.3** ([18, 7.1.11]) *If  $l(F_i)$  denotes the length of face  $F_i$  in a plane graph  $G$ , then  $2\varepsilon(G) = \sum_i l(F_i)$ .*

**Theorem 2.4** ([17]) *The smallest order of a triangle-free graph with  $\chi_1(G) = 3$  is 9, that is  $f(3, 1) = 9$ . Moreover,  $G$  is a triangle-free graph of order 9 with  $\chi_1(G) = 3$  if and only if it is isomorphic to one of the graphs  $G_i$ ,  $1 \leq i \leq 4$  given in Figure 1.*

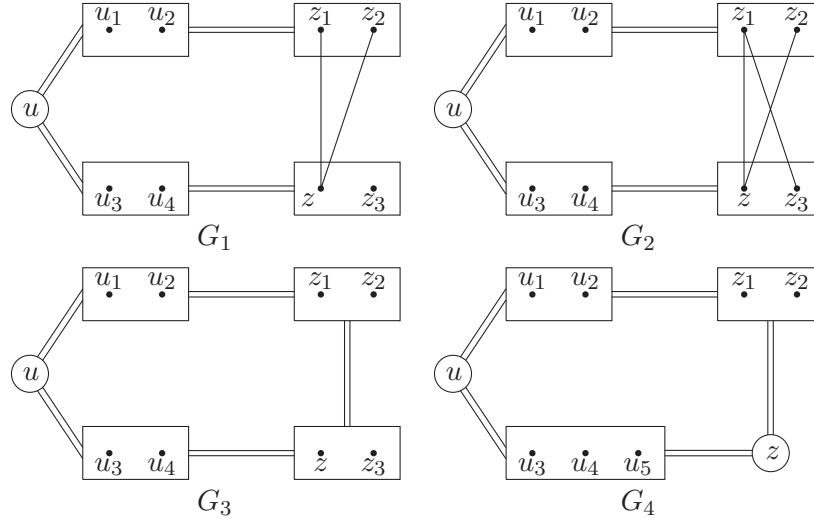


Figure 1: The critical graphs of order 9 with  $\chi_1(G) = 3$ :  $G_1$  to  $G_4$  of [17].

**Theorem 2.5** ([1]) *Let  $G$  be a triangle-free graph of order 10 with  $\chi_1(G) = 3$ . Then either  $G \cong G_5$  (Figure 2) or there exists a vertex  $u^*$  such that  $G - u^* \cong G_i$  for some  $i$ ,  $1 \leq i \leq 4$ .*

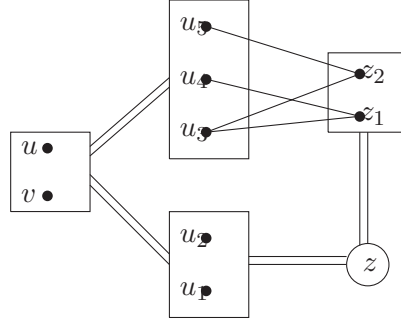


Figure 2:  $G_5$

The result of Theorem 2.5 can also be obtained by computations, using the methods described in the next section.

### 3 Computations

The result of Theorem 4.2 was first obtained by computations described here.

For convenience we use the abbreviations “tfp graph” for a triangle-free planar graph and “mtfp graph” for a maximal tfp graph, maximal in the sense that if another edge were to be added the graph so formed would either fail to be triangle-free or fail to be planar or both. Recall that  $\chi_k(G) \leq \chi_k(H)$  whenever  $G$  is a subgraph of  $H$ . Hence if  $G$  is a tfp graph with  $\chi_1(G) = 3$ , we can add edges to  $G$  to form a maximal tfp graph  $\tilde{G}$  with  $\chi_1(\tilde{G}) = 3$ .

To establish Theorem 4.2 (and rather more) all 11-vertex tfp graphs were generated using nauty [16]. These were then read into a computer algebra system and tested to find which ones were not  $(2, 1)$ –colourable.

The algorithm to generate a  $(2, 1)$ –colouring of a triangle-free graph is provided by the following Theorem.

**Theorem 3.1** *A graph  $G$  of order  $n$  is  $(2, 1)$ -colourable (i.e. has  $\chi_1(G) \leq 2$ ) if and only if the system of equations*

$$\begin{aligned} b_j^2 &= 1 && \text{for all } j \in V(G) \\ b_i b_j + b_j b_k + b_k b_i &= 2 && \text{for all paths } (i, j, k) \text{ of length 2} \end{aligned}$$

*has a solution in  $Z_3^n$  where  $Z_3$  is a finite field.*

*Proof.* In  $Z_3$ , with  $b_l^2 = 1$  for all  $l = i, j, k$ , we have,

$$b_i b_j + b_j b_k + b_k b_i = (b_i - b_j)^2 + (b_j - b_k)^2 + (b_k - b_i)^2.$$

Consider a  $(2, 1)$ -colouring of  $G$  using colours 1 and 2. We define  $b_i$  to be 1 or 2 according as the vertex  $i$  is assigned colour 1 or 2.

Now it is easy to see that precisely 2 of the 3 vertices on any path  $(i, j, k)$  are assigned the same colour. Thus the equations of Theorem 3.1 are satisfied. The converse is easy to establish.  $\square$

We remark that the  $(2, 1)$ -colouring problem can be formulated in a fashion similar to Theorem 3.1 but using systems of quadratics over  $Z_2$  rather than over  $Z_3$ . The connection with 3-SAT computation is given in [7, 9].

Solving, with  $n = 11$ , the system of equations given in Theorem 3.1 with respect to each of tfp graphs of order 11 generated by nauty, we found that there are exactly six graphs with 1-defective chromatic number equal to 3. These are presented as  $G_{pi}$ ,  $1 \leq i \leq 6$  in Figure 5. All these six graphs have exactly 17 edges and they are all mtfp graphs.

As mentioned above, Theorem 2.5 can be established computationally. This time one begins by using nauty to generate all 10-vertex triangle-free graphs, not merely planar ones. Solving, with  $n = 10$ , the system of equations given in Theorem 3.1 with respect to each of triangle-free graphs of order 10 generated by nauty, we establish Theorem 2.5.

## 4 Main results

We first prove the following useful lemmas.

**Lemma 4.1** *In any  $(2, k)$ -colouring of  $K_{2, \ell}$  with  $\ell \geq 2k + 1$  the two vertices of degree  $\ell$  must necessarily receive the same colour. Furthermore,  $K_{2, \ell}$  with  $\ell \geq 3$  is uniquely  $(2, 1)$ -colourable.*

*Proof.* Consider a  $(2, k)$ -colouring of  $K_{2, \ell}$ . Let  $C_1$  and  $C_2$  be the colour classes of this  $(2, k)$ -colouring. Without loss of generality let  $|C_1| \geq |C_2|$ . Clearly  $|C_1| \geq k + 2$ . If  $C_1$  contains exactly one vertex of degree  $\ell$ , then  $C_1$  is not  $k$ -independent. Hence either  $C_1$  contains both the vertices of degree  $\ell$  or contains neither vertex of degree  $\ell$ . Now if  $k = 1$ , clearly  $C_2$  must contain both the vertices of degree  $\ell$  and  $|C_2| = 2$ . This implies that the graph is uniquely  $(2, 1)$ -colourable. This establishes the lemma.  $\square$

The proof of Theorem 4.2 uses the above result with  $k = 1$  and  $\ell = 3$ .

**Lemma 4.2** *The graphs  $G_i$ ,  $1 \leq i \leq 5$  shown in Figures 1 and 2 are nonplanar.*

*Proof.* Note that the subgraph induced on the set  $\{u, u_1, u_2, u_3, z, z_1, z_2\}$  in both  $G_1$  and  $G_4$  is a subdivision of  $K_{3,3}$  as illustrated in (a) of Figure 3. Hence  $G_1$  and  $G_4$  are nonplanar.

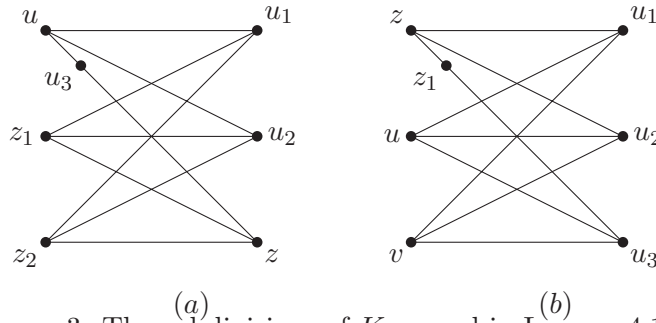


Figure 3: The subdivisions of  $K_{3,3}$  used in Lemma 4.1.

Since  $G_2 \cong G_1 + (z_1, z_3)$  and  $G_3 \cong G_2 + (z_2, z_3)$ ,  $G_2$  and  $G_3$  are also nonplanar. The subgraph of  $G_5$  induced on the set  $\{u, v, u_1, u_2, u_3, z, z_1\}$  is a subdivision of  $K_{3,3}$  as illustrated in (b) of Figure 3. Hence the graph  $G_5$  is also nonplanar.

This proves the lemma.  $\square$

**Lemma 4.3** *Let  $G$  be a maximal triangle-free planar (mtfp) graph of order  $n$  with at least one odd cycle. Then the number of edges of  $G$ ,  $\varepsilon(G)$  satisfies*

$$\varepsilon(G) = 2n - 4 - \frac{f_5}{2} \leq 2n - 5 \quad (1)$$

where  $f_5$  is the number of faces with 5 edges in any planar embedding of  $G$ .

*Proof.* Let  $C$  be a shortest odd cycle and let  $\ell$  be the length of  $C$ . Note the  $C$  is chordless. If  $\ell \geq 7$  then we can add a chord without creating a  $K_3$  or destroying the planarity of  $G$ . This contradicts the maximality of  $G$ . Hence  $\ell = 5$ .

Consider a planar embedding  $G'$  of  $G$ . Clearly  $\varepsilon(G') = \varepsilon(G)$  and every face of  $G'$  is of length 4 or 5. Let  $f_i$  be the number of faces with  $i$  edges,  $i = 4$  or  $5$ . From [18, 7.1.13] it follows that  $f_5 > 0$ .

Using Euler's formula we have

$$n - \varepsilon(G') + f_4 + f_5 = 2. \quad (2)$$

Now using Theorem 2.3,

$$4f_4 + 5f_5 = 2\varepsilon(G'). \quad (3)$$

Elimination of  $f_4$  from equations (2) and (3) yields the equation at the left of (1). From this it follows that  $f_5$  is even. Since  $f_5 > 0$  we have  $f_5 \geq 2$ . This then yields the inequality at the right of (1), and completes the proof.  $\square$

We remark that inequality (1) remains valid for any tfp graph of order  $n$  with at least one odd cycle.

**Theorem 4.1** *Let  $G$  be a maximal triangle-free planar graph of order 11 with an odd cycle. Then  $15 \leq \varepsilon(G) \leq 17$ .*

*Proof.* Consider a planar embedding  $G'$  of  $G$

Since  $G'$  contains odd cycles, from Lemma 4.3 we have  $\varepsilon(G') = 18 - \frac{f_5}{2} \leq 17$ . Since  $f_4 \geq 0$  we have

$$5f_5 \leq 4f_4 + 5f_5 = 2\varepsilon(G) = 36 - f_5.$$

Thus  $6f_5 \leq 36$  implying that  $f_5 \leq 6$ . Since  $f_5 > 0$  and even,  $f_5 = 2, 4$  or  $6$ . Thus  $\varepsilon(G) = 15, 16$  or  $17$  according as  $f_5 = 6, 4$  or  $2$ . This completes the proof.  $\square$

We now present the main result of the paper.

**Theorem 4.2** *The smallest order of a triangle-free planar graph  $G$  with  $\chi_1(G) = 3$  is 11, that is,  $f(3, 1; \text{planar}) = 11$ .*

*Proof.* From Theorem 2.5 and Lemma 4.2, it is easy to see that there is no planar triangle-free graph  $G$  on 10 or fewer vertices with  $\chi_1(G) = 3$ . Hence we have  $f(3, 1; \text{planar}) \geq 11$ . To establish equality, consider the tfp graph  $G_{p1}$  of order 11 shown in Figure 4.

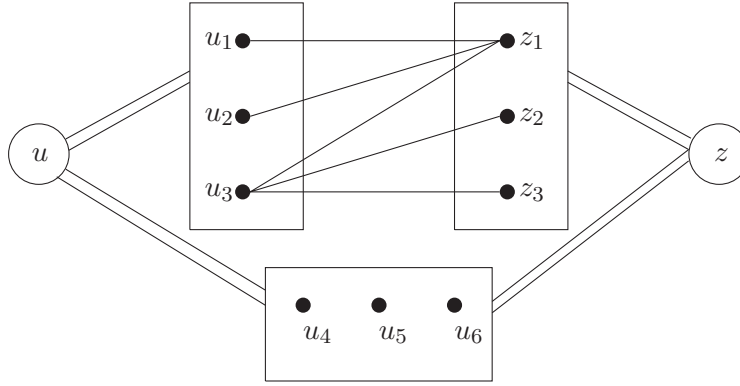


Figure 4: The graph  $G^*$  used in Theorem 4.2.

Suppose  $G_{p1}$  is  $(2, 1)$ -colourable and consider a  $(2, 1)$ -colouring of  $G_{p1}$  with colours 1 and 2. Without loss of generality, we assume that vertex  $u$  is assigned colour 1. Apply Lemma 4.1 three times.

- (i) Applying it first to  $G_{p1}[\{u, u_1, u_2, u_3, z_1\}] = K_{2,3}$  we have that  $z_1$  is coloured 1 and hence  $u_3$  is assigned colour 2.
- (ii) Now applying Lemma 4.1 to  $G_{p1}[\{u_3, z_1, z_2, z_3, z\}] = K_{2,3}$  it follows that  $z$  is assigned colour 2.
- (iii) Finally applying Lemma 4.1 to  $G_{p1}[\{u, u_4, u_5, u_6, z\}] = K_{2,3}$  it follows that  $z$  is assigned colour 1.

But items (ii) and (iii) above are contradictory and so  $G_{p1}$  is not  $(2, 1)$ -colourable. It is easy to see that  $G_{p1}$  is  $(3, 1)$ -colourable (and in fact,  $(3, 0)$ -colourable by Grötzsch theorem). Hence  $\chi_1(G_{p1}) = 3$  and this completes the proof of Theorem 4.2.  $\square$



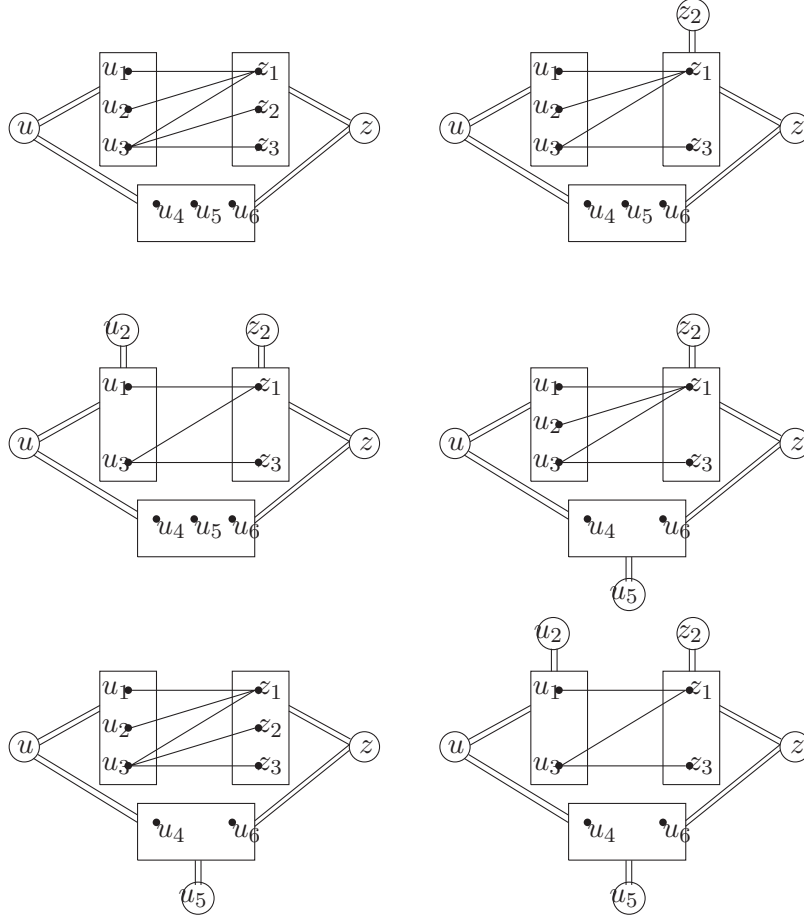


Figure 5: The 6 graphs  $G_{pi}$  for  $1 \leq i \leq 6$ .

The graph  $G_{p6} - (u_3, z_1)$  is isomorphic to the “cross-over” graph of Figure 3 of [7].

**Lemma 4.4** *Let  $G$  be a tfp graph of order 11 with  $\delta(G) = 2$ . Let  $e$  be an edge incident with a vertex of degree 2. Then  $\chi_1(G - e) = 2$ .*

*Proof.* Suppose that  $e = (x, y)$  and  $d_G(x) = 2$ . Let  $x_1$  be the other neighbour of  $x$ . The graph  $G - x$  is tfp graph of order 10 and hence  $\chi_1(G - x) = 2$ . Consider a  $(2, 1)$ -colouring of  $G - x$  using colours 1 and 2. Without loss of generality assume that the vertex  $x_1$  is assigned colour 1. Now assign

colour 2 to vertex  $x$ . This provides a  $(2, 1)$ –colouring of  $G - e$  and hence  $\chi_1(G - e) = 2$ . This completes the proof of Lemma 4.4.  $\square$

The result of the next lemma was evident from the computations, but here we give a simple direct proof.

**Lemma 4.5** *The tfp graph  $G_{p6}$  of order 11 given in Figure 5 is  $(3, 1)$ –edge-critical.*

*Proof.* Let  $e = (x, y)$  be an edge of  $G$ . By Lemma 4.4, we can assume that  $d_G(x) > 2$  and  $d_G(y) > 2$ . There are precisely 11 such edges in  $G_{p6}$  and they are one of five types. For a representative edge  $e = (x, y)$  of each type the table below describes a  $(2, 1)$ –colouring of  $G - e$ . The  $(2, 1)$ –colouring is defined by the partition of  $V(G)$  into 1–independent sets  $X(e)$  and  $V(G) - X(e)$ .

$e$	$X(e)$	other edges of same type
$(u, u_4)$	$\{u, z_1, z_3, u_2, u_4, u_6\}$	$(u, u_6), (z, u_4), (z, u_6)$
$(u, u_1)$	$\{u, z, u_1, u_3, u_5, z_2\}$	$(z, z_3)$
$(u_1, z_1)$	$\{u, z, u_3, u_5, z_2\}$	$(u_3, z_3)$
$(u, u_3)$	$\{u, z, u_1, u_3, u_5, z_2\}$	$(z, z_1)$
$(u_3, z_1)$	$\{u, z, u_2, u_5, z_2\}$	

This establishes the Lemma 4.5.  $\square$

**Theorem 4.3** *The tfp graphs  $G_{pi}$  for  $1 \leq i \leq 6$  are  $(3, 1)$ –edge-critical.*

*Proof.* The Lemma 4.5 establishes the result for graph  $G_{p6}$ . The results for the other  $G_{pi}$  are proved in a similar manner. These cases are simpler as for each of them the number of edges free of degree 2 is smaller than it was for the graph  $G_{p6}$ . More precisely there are just 3, 5, 5, 9 and 7 such edges respectively for the graphs  $G_{pi}$ ,  $1 \leq i \leq 5$ .  $\square$

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